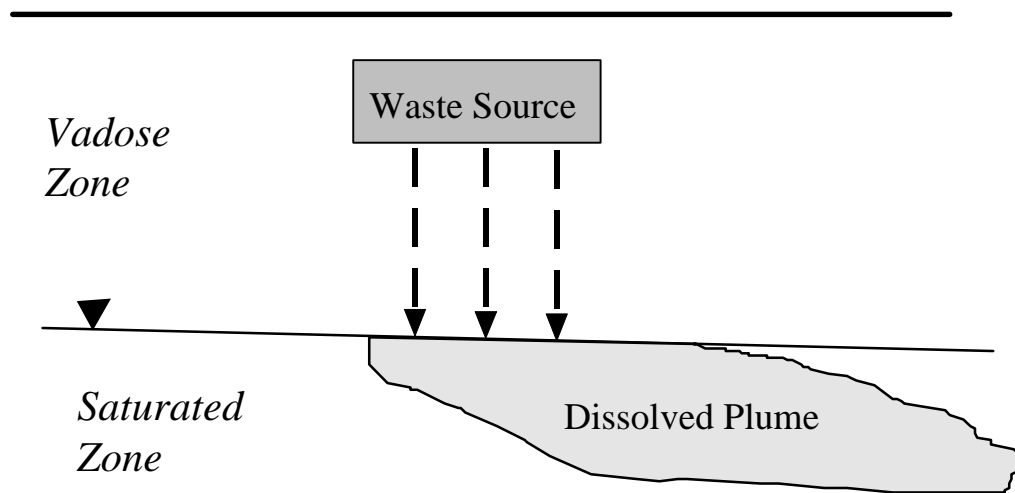# UNSATURATED FLOW:

WHY DO WE CARE?

Source term for contaminants entering saturated zone from contaminants placed in the unsaturated zone.

Need to know mass flux of contaminants to saturated zone to model contaminant transport in groundwater.

Contaminants in vadose zone transported as:
- dissolved constituents in infiltrating water (advection, dispersion, diffusion)
- vapor in air phase (diffusion)

Supplementary Reading:
Bear: Ch. 9.4
Bedient, Rifai, and Newell: Ch. 9
Corey: Ch. 1-5

UNSATURATED FLOW EQUATIONS:

Assume: Incompressible Fluid (r = constant)
            Nondeformable Media (n = constant)
            Air does not restrict water from filling void space
            Principle axes aligned with media anisotropy

Continuity Equation (volumetric basis)

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z}$$

Note that: $\theta = nS$

Since porosity, n, is constant, water content can change only when saturation changes.

The Buckingham Equation:
      - Extends the Darcy equation to unsaturated flow.
      - Hydraulic conductivity, K, is not constant!
            $K = f(\theta) = g(\Psi)$

Remember: Hydraulic head = pressure head + elevation head

$$h = \frac{P}{\rho g} + z = \Psi + z$$

Ψ = pressure head, or matric potential, or capillary head

UNSATURATED FLOW EQUATIONS: *(continued)*

Buckingham equation:

$$q_x = -K_x(\theta)\frac{\partial h}{\partial x} = -K_x(\Psi)\frac{\partial h}{\partial x}$$

$$q_y = -K_y(\theta)\frac{\partial h}{\partial y} = -K_y(\Psi)\frac{\partial h}{\partial y}$$

$$q_z = -K_z(\theta)\frac{\partial h}{\partial z} = -K_z(\Psi)\frac{\partial h}{\partial z}$$

Insert into the continuity equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}K_x(\Psi)\frac{\partial h}{\partial x} \quad + \quad \frac{\partial}{\partial y}K_y(\Psi)\frac{\partial h}{\partial y} \quad + \quad \frac{\partial}{\partial z}K_z(\Psi)\frac{\partial h}{\partial z}$$

Rewrite equation in terms of pressure head, Ψ :

$$\frac{\partial \theta}{\partial \Psi}\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial x}\left[K_x(\Psi)\left(\frac{\partial \Psi}{\partial x}+\frac{\partial z}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[K_y(\Psi)\left(\frac{\partial \Psi}{\partial y}+\frac{\partial z}{\partial y}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[K_z(\Psi)\left(\frac{\partial \Psi}{\partial z}+\frac{\partial z}{\partial z}\right)\right]$$

UNSATURATED FLOW EQUATIONS: *(continued)*

Note that Specific Moisture Capacity, $C(\Psi) = \dfrac{\partial \theta}{\partial \Psi}$

Governing Equation for Unsaturated Flow = Richard's Equation:

$$C(\Psi)\frac{\partial \Psi}{\partial t} \;=\; \frac{\partial}{\partial x}\left[K_x(\Psi)\frac{\partial \Psi}{\partial x}\right] \;+\; \frac{\partial}{\partial y}\left[K_y(\Psi)\frac{\partial \Psi}{\partial y}\right] \;+\; \frac{\partial}{\partial z}\left[K_z(\Psi)\left(\frac{\partial \Psi}{\partial z}+1\right)\right]$$

Major Differences Between SATURATED and TRANSIENT UNSATURATED Flow in Porous Media:

Parameter	Saturated	Unsaturated
Water content (θ)	$\theta = \theta_{sat} = n$	$0 < \theta < n$
Total potential, h	$h = \dfrac{P}{\rho g} + z \quad (P \geq 0)$	$h = \Psi + z \; (-\infty < \Psi < 0)$
Dominant force	Gravity (z)	Matric Potential (Ψ)
Driving Force	$\partial h/\partial x \cong$ order 1	$\partial h/\partial x \cong$ order $(1-10^4)$
Hydraulic Conductivity (K)	$K = K_{sat}$	$K = K(\theta)$ or $K(\Psi)$ Range 3 - 8 orders of magnitude
Flux (q)	$q = -K_{sat}\,\partial h/\partial x$	$q = -K(\Psi)\partial h/\partial x$
Governing flow equation	Laplace Equation $\nabla^2 h = 0$	Richards Equation $\dfrac{\partial \theta}{\partial t} = \nabla\left[K(\Psi)\nabla h\right]$

INTERFACIAL TENSION - CAPILLARY PRESSURE

Consider the interface between two fluids (water and air):

- water molecules @ interface are more strongly attracted to water than air.
- energy req'd to maintain molecules @ interface = interfacial (surface) tension.
- results in curved interface .
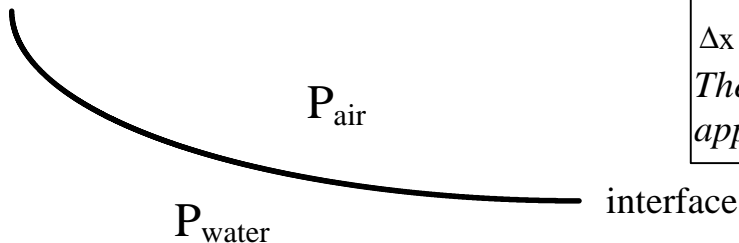
> *Finite difference between pressures in air and water at the interface:*
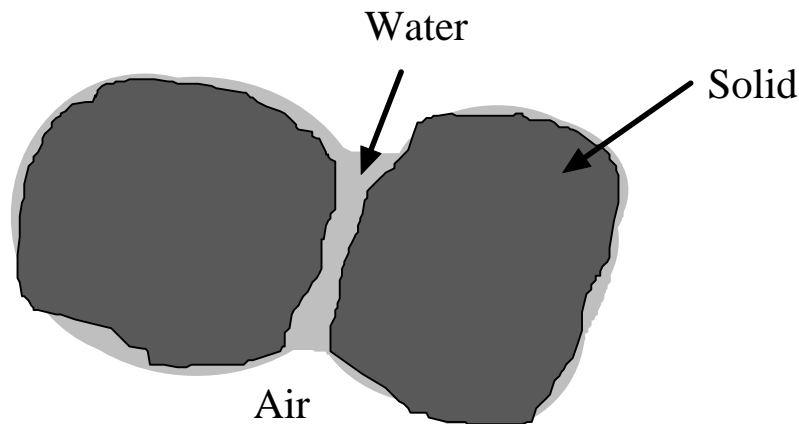> $$\lim_{\Delta x \to 0} \frac{P_{air} - P_{water}}{\Delta x} = \infty$$
> *Therefore must apply continuum approach to each phase separately.*

P_{air}

P_{water}

interface

Capillary Pressure $= P_c = P_{NW} - P_W$

NW = nonwetting phase (air in air/water system)
W = wetting phase (water in air/water system)

Wetting Phase = Fluid phase that preferentially coats the solid phase.
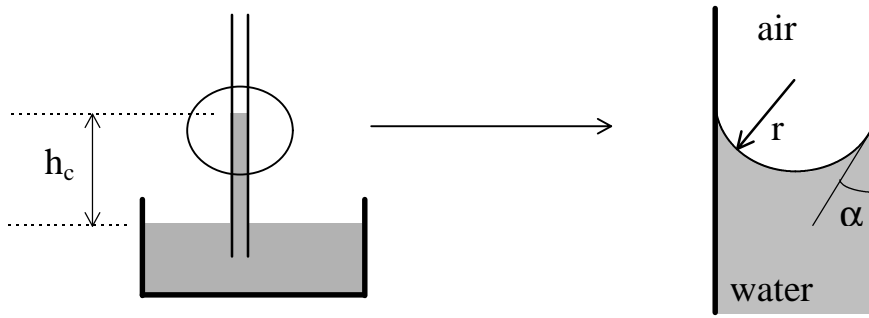
Water

Solid

Air

Relationship between capillary pressure and matric potential:

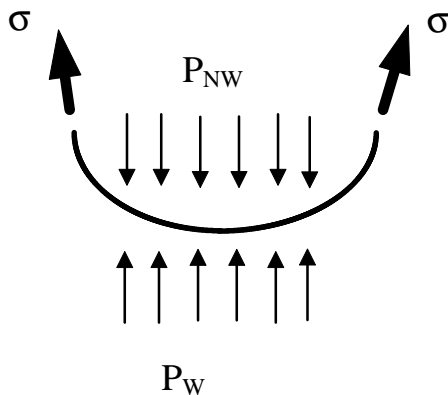$$P_c = P_{NW} - P_W = P_{air} - P_{water} = -P_{water}$$

$$P_{air} = P_{atm} = 0$$

$$\Psi = \frac{P_w}{\rho g} = -\frac{P_c}{\rho g}$$

Consider water rise in a capillary tube:

Balance Interfacial Tension and Pressure forces:

$$\left(P_{NW} - P_W\right)\pi r^2 = 2\pi r \sigma_{lg} \cos\alpha$$
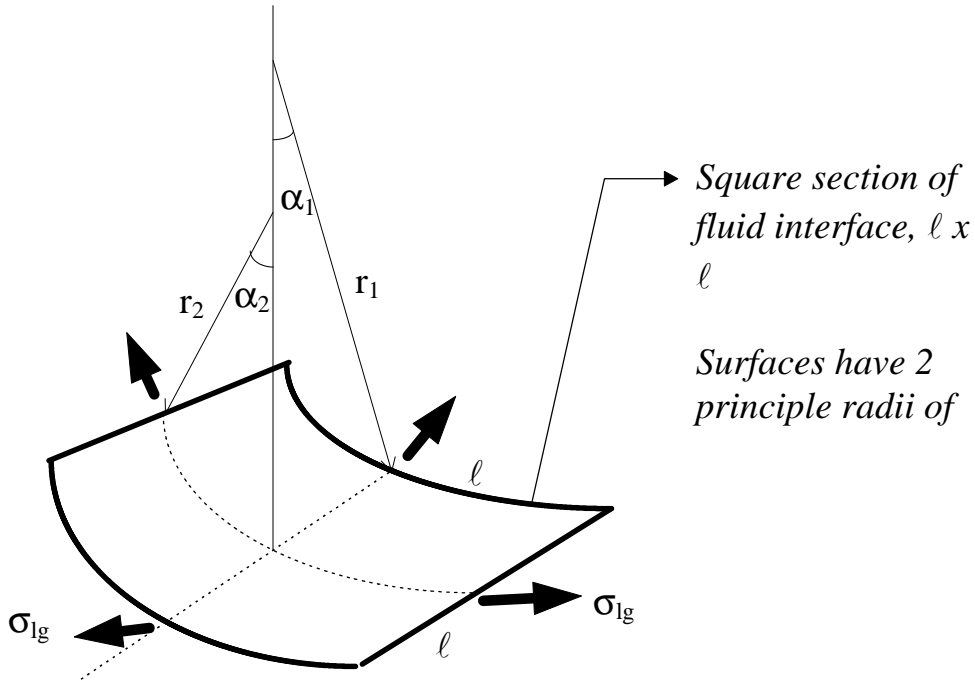
$$P_c = \frac{2\sigma_{lg}}{r} \cos\alpha$$

Capillary rise in a tube:

$$h_c = \frac{P_c}{\rho g}$$

Capillary Pressure - Interfacial Tension (continued):

More general case: LaPlace Equation of Capillarity

Square section of fluid interface, ℓ x ℓ

Surfaces have 2 principle radii of

$$P_c \ell^2 = 2\sigma_{lg} \ell \left(\sin\alpha_1 + \sin\alpha_2 \right)$$

if r_1, $r_2 \gg \ell$ (α_1, α_2 are small); $\sin\alpha \cong \dfrac{\ell}{2r}$

$$P_c = \sigma_{lg} \left(\frac{1}{r_1} + \frac{1}{r_2} \right)$$

Convention: $r > 0$ if radius is in nonwetting phase.
                  $r < 0$ if radius is in wetting phase.

Values of Interfacial Tension:

Fluid 1	Fluid 2	σ (dynes/cm^2) (20°C)
Pure water	Air	72
Soil water	Air	~60
Soltrol (petroleum)	Air	~23
Mercury	Air	470
Mercury	Water	375
Water	Octane	51
Ethanol	Air	21.5
Water	Octanol	8.5
Water	Alcohol	0 (miscible)